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An increase in the size of coherent domains in the one component Φ^4 field theory under the influence of a uniformly changing external magnetic field near the critical end-point $T_\Phi = T_c, h_\Phi = 0$ was proposed recently as an estimate also for the variation of the chiral correlation length of QCD near its respective hypothetical end point in the $T_{QCD} - \mu_{QCD}$ plane. The present detailed numerical investigation of the effective model suggests that passing by the critical QCD end point with realistic rate of temperature change will trigger large amplitude oscillations in the temporal variation of the chiral correlation length. A simple mechanism for producing this phenomenon is suggested.

I. INTRODUCTION

The quality of our knowledge of the phase structure of QCD at high temperature and finite baryon density is an important benchmark for our understanding of strong interactions.

A critical end-point of the first order phase transition line in the $T - \mu$ -projection of the QCD phase diagram was conjectured [1] to follow from the compatibility of the following observations:

- i) Lattice simulations [2] indicate that the phase transformation at zero chemical potential with realistic quark masses is a crossover, characterised by an analytic variation of the thermodynamical potential with the temperature;
- ii) At zero temperature there is a first order phase transition from the hadron phase to more exotic phases as a function of μ [3] which continues as a first order line into the (T, μ) -plane.

For the strong matter the significance of this end-point would be similar to that of the Curie point for ferromagnets.

At present no one can predict the exact (μ_E, T_E) location of the end-point. There is a chance in current heavy ion collision experiments that it can be observed experimentally, since with increasing collision energy/nucleon the central rapidity particle spectra explore regions of the phase diagram corresponding to decreasing chemical potential. An obvious class of the signatures would reflect the increasing size of the coherent fluctuations in the order parameter σ -field in the neighbourhood of the end-point [4]. In this region mostly the σ -field will be excited, since its mass is the lightest near the critical end-point in view of the amount of explicit chiral symmetry breaking which keeps the pions massive. This coherence should be reflected by the statistics of the main decay products of the σ -field, the pions.

Starting from equilibrium at some $T_0 > T_c$, and passing with finite velocity near the end point, the system unavoidably slows out of equilibrium. In contrast to the equilibrium characterisation of the second order transition, finite maximal correlation length is reached with a certain amount of supercooling. A substantial increase in the correlation length will be a reliable signal for the existence of the critical end-point.

Quantitative predictions for this phenomenon should rely on non-equilibrium field theory. For the moment it is hopeless to simulate directly the far from equilibrium behaviour of QCD matter. One recognises however, that the mass of the σ in this region is separated by a gap from the other mesonic excitations. Therefore, one arrives at the conclusion that an effective dynamical theory of the longest wavelength excitations in this region is in the same universality class as the Ising model. At present no quantitative matching is known between the original theory and its one-component Φ^4 effective version. Still, universal features of the class A of critical dynamics (in the classification of Hohenberg and Halperin [5]) are expected to occur.¹

Recently Berdnikov and Rajagopal (BR) [6] proposed an intuitive mapping. They approximately identify the magnetic field (h) of the Ising model with the temperature of the QCD (the Ising reduced temperature (r) axis is nearly parallel to the chemical potential axis of QCD). They have checked that the results are not sensitive to a moderate tilt in this mapping. Next, they have proposed a dynamical equation which describes the evolution of the inverse correlation length (the mass of the σ meson) when the system passes through the critical end point of the Ising model with finite velocity under different angles. This equation was shown to depend on a single non-universal parameter, proportional to the rate of change of the Ising magnetic field.

Finally, a semi-quantitative correspondence was proposed between the relevant QCD temperature range ($T_0 = 180$ MeV, $T_{freezeout} = 120$ MeV) and the dimensionless strength of the Ising magnetic field ($h_0 = -0.2, h_{final} = 0.1$).

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¹A more complete theory reflecting the direct influence of the temporal variation of a conserved baryon number density on the σ field falls into class C [6], and will be subject of future study.

The non-universal constant was varied in a wide range, since it relates essentially the rate of cooling of the QCD-matter to the speed of the variation of the external field h . The main result of Ref. [6] was a prediction for the variation of the order parameter (e.g. σ) correlation length with the variation of h in the neighbourhood of the critical end-point.

The variation of the correlation length during the passage through the critical point was compared to its value taken at the non-critical starting value, which corresponds to the equilibrium energy density of the QCD plasma produced in the collision ($T = T_0$). It is the relative increase in the correlation length at the freeze-out of the system ($T = T_{freezeout}$), where most of the pions are coming from, which is the most important issue. A factor of 2-3 increase was signalled, which is claimed to be observable under the accuracy of the present heavy ion experiments.

The aim of our present investigation is to check the accuracy of the first order relaxational dynamics assumed for the inverse correlation length. For this we have integrated exactly the equations of motion of the one-component classical three-dimensional scalar field, $\Phi_d(\mathbf{x}, t)$ on lattices of size $N = 16 - 128$:

$$\ddot{\Phi}_d(\mathbf{x}, t) = (\Delta - m^2) \Phi_d(\mathbf{x}, t) - \frac{\lambda}{6} \Phi_d^3(\mathbf{x}, t) - h_d(t), \quad (1)$$

and measured simultaneously the variation of the order parameter and of the correlation length.

Though the hadronic system freezes out at $T_{freezeout} \sim 120$ MeV which corresponds to $h = 0.1$ in the effective system, we have followed the variation of the correlation length and of the order parameter to higher values of h . This enabled us to recognize the relevance of a second order dynamics in the effective equation of motion of these characteristics.

It turns out that the order parameter (OP) obeys an equation which is slightly more complicated than the one proposed by Halperin and Hohenberg for this class. It is formally analogous to the differential equation of a damped oscillator. In order to achieve good quantitative description of the OP-trajectory obtained from the numerical simulations, one has to take into account the effect of slowing out from equilibrium while the system passes by the critical point.

Our paper is organized the following way. In Section II the method of the numerical solution of (1) is shortly outlined. The methods of analyzing the time evolution of the system are presented in more detail with special emphasis on the determination of the spatial correlation length. In Section III the trajectories of the most important quantities characterizing the state of the system are mapped out in the (r, h) - and the (OP, h) -planes when passing at different distances by the critical Ising end point. In Section IV we present the results for the non-equilibrium h -evolution of the correlation length for several values of $a^{-1} \equiv dh/dt$ and compare them with the estimates of BR. Finally, in Section V a quantitative phenomenological interpretation of the measured order parameter trajectory is offered. Our dynamical description is compared to the equation of motion proposed for this class by Halperin and Hohenberg. Based on the proposed effective relaxational dynamics we suggest also a simple way to account for the variation of the correlation length. Conclusions of our investigation are summarized in Section VI.

II. METHODS OF SOLVING AND ANALYZING THE EXACT CLASSICAL EQUATIONS

For setting up and solving numerically the Φ^4 theory we used techniques similar to those applied in our previous paper [7]. Here we shortly outline the procedure.

As a first step one has to rewrite Eq.(1) using dimensionless quantities (not having index d), defined as follows:

$$\begin{aligned} t &= t_d/a_x, & x &= x_d/a_x, \\ \Phi &= \sqrt{\frac{\lambda}{6}} a_x \Phi_d, & h &= \sqrt{\frac{\lambda}{6}} a_x^3 h_d, \end{aligned} \quad (2)$$

where a_x stands for the lattice spacing, the powers of which we used to scale dimensionfull quantities. The dimensionless mass parameter of the theory $|m| = a_x |m_d|$ was set to unity. The field equation in dimensionless variables is of the following form:

$$\Phi_{\mathbf{n}}(t + a_t) + \Phi_{\mathbf{n}}(t - a_t) - 2\Phi_{\mathbf{n}}(t) - a_t^2/a_x^2 \sum_i (\Phi_{\mathbf{n}+\hat{i}}(t) + \Phi_{\mathbf{n}-\hat{i}}(t) - 2\Phi_{\mathbf{n}}(t)) + a_t^2(-\Phi_{\mathbf{n}} + \Phi_{\mathbf{n}}^3 - h(t)) = 0. \quad (3)$$

There were a number of simple quantities routinely monitored in each run. The first was twice the average kinetic energy per site (called kinetic temperature also for non-equilibrium field configurations):

$$T_{kin}(t) = \frac{1}{L^3} \sum_{\mathbf{n}} \dot{\Phi}_{\mathbf{n}}^2(t), \quad (4)$$

the second, the trajectory of the homogeneous (OP) mode:

$$\bar{\Phi}(t) = \frac{1}{L^3} \sum_{\mathbf{n}} \Phi_{\mathbf{n}}(t) \equiv \overline{\Phi(x, t)}^V. \quad (5)$$

$$\delta\Phi^2(t) = \overline{\Phi^2(x,t)}^V - \left(\overline{\Phi(x,t)}^V\right)^2 \quad (6)$$

was used.

The thermalization algorithm, which led to the initial state, consisted of two steps. First we have set the initial energy density by continuously comparing the desired and the measured kinetic temperatures. Depending on the deviation from the targeted temperature an artificial friction or anti-friction term was introduced into Eq.(3). After having reached the required kinetic energy density, in a second step the original form of the discretized nonlinear field equation was iterated until thermalization was complete. Both steps were repeated until the final kinetic temperature was just what we desired.

The critical temperature $T_{\Phi,c}$ was determined at $h = 0$ by locating on the T_{Φ} axis the maximum of $\delta\Phi^2$, or of the specific heat, and by finding the point separating zero and non-zero expectation value regions of OP, $\bar{\Phi}$. The reduced temperature $r \equiv (T_{\Phi} - T_{\Phi,c})/T_{\Phi,c}$ was measured relative to $T_{\Phi,c} = 1.5$ for $L = 32$ and $T_{\Phi,c} = 1.57$ for $L = 64$.

A. Determination of the correlation length

The main goal of this Paper is to give a quantitative interpretation to the dynamical behaviour of the correlation length as the system passes by the critical end point. Therefore we need an accurate measurement method for this quantity, which is reliable in a dynamical system too.

1. The most straightforward way is to use the definition of the correlation function:

$$C(\Delta, t) = \overline{\langle \Phi(x, y, z + \Delta, t) \Phi(z, y, z, t) \rangle} - \langle \bar{\Phi}(t) \rangle^2. \quad (7)$$

Here the overline refers to the spatial average of some quantity at fixed t in a single sample, while the brackets stand for the average over the initial conditions. After checking that $C(\Delta, t)$ truly behaves as $\sim \cosh[(\Delta - L/2)/\xi_1]$, one can extract the correlation length ξ_1 . We refer to this characteristic length as the “direct” length of correlation below.

2. For the characterisation of the average size of coherently developing domains we classified the lattice sites of a configuration at given time into two classes: the first where $\Phi(\mathbf{x}, t) > \bar{\Phi}(t)$ and the other where $\Phi(\mathbf{x}, t) < \bar{\Phi}(t)$. On this Ising-like map we counted the number $N(\Delta, t)$ of how many times sequences of lattice points belonging to the same class, parallel to one axis and of length Δ are encountered. This histogram shows a perfect exponential dependence on Δ . The characteristic length calculated from the histogram is related to the average size of the coherent domains. We shall refer below to the average of this quantity over the initial conditions as the average “grain size” size ξ_2 at time t .

The correlation lengths defined by the above algorithms are different, of course. One expects, however, that both definitions capture the same feature of a field configuration and there exists a simple functional relationship between them. In order to find it we normalize both to a more standard quantity, which – in equilibrium – equals with certainty to the mass. We have measured in equilibrium systems for several values of the reduced temperature $r \in 0..0.27$ at $h = 0$ both the mass $m(r)$ using the algorithm 5 (see below) and the above defined characteristic lengths ξ_i . Assuming the $m = g_i(\xi_i)$ relations obtained in this way to be valid in an arbitrary point of the (r, h) -plane one obtains the normalisation of the ξ_i lengths. We found that the corresponding functions g_i may be well approximated by second order polynomials. These simple functions were used also for out-of-equilibrium states.

B. Spectral determination of the mass

An algorithm for the reconstruction of the effective potential from the real time evolution of a scalar field was presented in [7]. An effective equation of motion was fitted to the temporal variation of the order parameter. Its “force” term was interpreted as the derivative of the effective potential with respect to the field. Then it was easy to identify the effective squared mass.

This time we improved further this algorithm. All spatial Fourier modes of the system are used for the reconstruction of the dispersion relations. Three stages of the implementation were worked out.

3. We assumed the validity of the following approximate equation for short time intervals:

$$Z\ddot{\Phi} - \Delta\Phi + m_{eff}^2\Phi + \epsilon\Delta^2\Phi = 0, \quad (8)$$

where Δ is understood as lattice discretization to the Laplacian. Comparing this to the real nonlinear evolution we could fit Z, m_{eff}^2, ϵ for the short intervals in question in the following way. A spatial FFT algorithm was applied to the field configurations generated by Eq.(3). The temporal trajectory of each $\hat{\mathbf{k}}$ mode was fitted to the equation dictated by the Fourier transform of Eq.(8). The coefficients of the polynomial of \hat{k}^2 determine also the time dependent effective mass. (Here and in the following $\hat{k}_i = 2(\sin k_i/2)$ stands for the dimensionless lattice momentum.) A lower bound for the length of time interval in which Eq. (8) is fitted comes from the consistency criterion that the time interval we averaged over must be much greater than the resulting $1/m_{eff}$ time scale. For out-of-equilibrium field configurations there is also an upper bound for the time interval which is set by the variation rate of the parameters r, h .

4. For the two-point function related to the OP-susceptibility [8] the relation

$$\frac{T}{\langle |\Phi_{\mathbf{k}}|^2 \rangle} = Z^{-1} \left(m_{eff}^2 + \hat{k}^2 + \epsilon \hat{k}^4 + \mathcal{O}(\hat{k}^6) \right) \quad (9)$$

holds in equilibrium [9]. Replacing the ensemble average in the denominator of the left hand side by averaging over the \mathbf{k} -modes characterized by equal \hat{k}^2 and using $T_{kin, \hat{k}^2} = \overline{|\Phi_{\mathbf{k}}|^2}^{\Omega_{\mathbf{k}}}$ for T we found that our system obeys Eq.(9) for $\hat{k}^2 > 1$ without any time averaging. With these replacements we could fit the value of $m_{eff}^2(h)$ continuously both near and far from equilibrium. The wave function renormalization constant was found to be equal to 1 up to 0.5 percent in all cases. The coefficient ϵ of the fourth derivative correction term was checked to be negligible.

5. Since most of modes with $\hat{k}^2 > 1$ nicely follow the above behaviour, one might improve the statistical confidence of the above algorithm by summing over the contribution of all modes (including $\hat{k}^2 < 1$), which yields the approximate equality

$$\frac{\delta\Phi^2}{T} = \frac{1}{L^3} \sum_{\mathbf{k} \neq \mathbf{0}} \frac{1}{m_{eff}^2 + \hat{k}^2}, \quad (10)$$

with $\delta\Phi^2$ denoting the field fluctuation as defined in Eq.(6). Its measurement does not require the application of the time consuming FFT. The value of the function on the right hand side can be tabulated and Eq.(10) is quickly solved with help of a look-up table plus interpolation for every measured value of $\delta\Phi^2$.

In addition, as a quick reference the so-called Hartree squared mass estimate $m_{Hartree}^2 = -1 + 3(\bar{\Phi}^2 + \delta\Phi^2)$ was also used. It proved useful in interpreting qualitatively effects in the motion of modes with different $\hat{\mathbf{k}}$ and apparently related to the presence of an h -dependent common effective mass.

After careful testing of the simplified procedures against the conceptually better funded methods in equilibrium we decided to use Algorithm 5 throughout this paper.

Although the different algorithms were normalized to yield equal masses in equilibrium, it is not obvious that they will agree also for a non-equilibrium crossover connecting two points of the (r, h) plane. We shall return to the comparison of the non-equilibrium results obtained with different methods at the end of section IV.

III. PASSING BY THE CRITICAL POINT

We have solved Eq.(3), the discretized version of Eq.(1) numerically. A starting configuration was thermalized in presence of the initial magnetic field ($h_0 = -0.2$), in such a way that a predefined value of the average kinetic energy density was reached. This thermal state was taken for the initial configuration when solving the field equation with an external magnetic field tuned with constant velocity dh/dt . With the parameters we used the correlation length of the initial configuration was approximately equal to one lattice unit.

In the present investigation the same range of the parameters $r, dh/dt$ was covered as in [6]. For each value of $r, L, a \equiv (dh/dt)^{-1}$ runs with ~ 20 different equilibrium configurations were averaged.

An approximate idea about the part of the Ising phase diagram explored in our numerical investigation can be given by drawing the measured (r, h) trajectories for different values of $r_{initial}$ and a (see Fig.1).

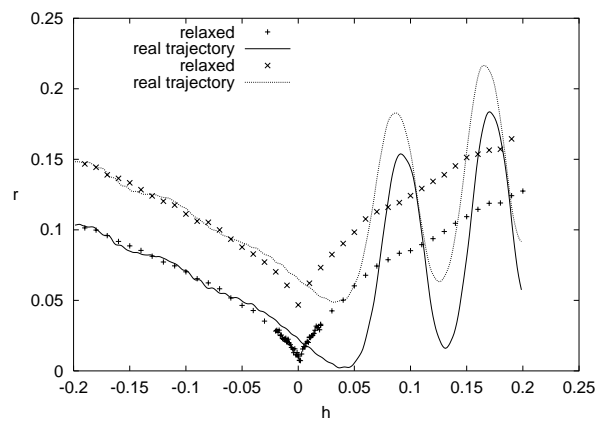


FIG. 1. The measured trajectory in the (r, h) -plane for two different initial temperatures (solid lines). The crosses show the reduced equilibrium temperature $r(h)$ corresponding to energy density $\epsilon(h)$. The size of the system was $L = 64$, and the inverse rate of change of h : $a = 100$.

Due to the time dependence of the external force $h(t)$ the full energy density $\epsilon(t)$ is not conserved, we shall parametrize it as $\epsilon(h)$. In principle, it might depend on the parameter a too, but for its range investigated in this paper, we did not experience any a -dependence of the energy density. The reduced (Ising) temperature, which is defined through the average kinetic energy per site, slowly drops in the first part of the crossover. However, having passed by the critical point ($h = 0, r = 0$) a deterministic oscillation in r starts, which survives the averaging over the ensemble of initial thermal configurations. For comparison we have displayed in Fig.1 also the reduced temperatures corresponding to the equilibrium states belonging to the different values of the energy density $\epsilon(h)$. The latter was determined by stopping the variation of h at a certain value h_{stop} , and then relaxing the system with fixed $h = h_{stop}$ (and hence, with conserved energy) to equilibrium. After the thermalization of this state was complete, we have measured its equilibrium temperature. These runs were performed for every non-equilibrium trajectory for 40 equidistant intermediate values of $h_{stop} \in (-0.2, 0.2)$.

Fig.1 clearly shows how the system slows out of equilibrium. As the critical point is approached, the thermalization time scale grows above the time scale ($\sim ha$) of moving in the phase diagram. Thus, in the second part of the crossover two effects seem to be present: First, the spectral density of the configuration will correspond to the equilibrium at an earlier h value – this means an overcooling in QCD language. Second, the low- \mathbf{k} modes get highly excited and begin to oscillate with a frequency determined by an effective mass scale. The oscillation is synchronously present in the entire spectrum, i.e. the kinetic energy of each “well-behaving” ($\hat{k}^2 > 1$) mode oscillates coherently. The oscillation is not due to any direct strong coupling between the modes, but is driven by the oscillation of the homogeneous mode ($\bar{\Phi}$) (see Fig 2a). The UV modes adiabatically follow the slow oscillation of the order parameter (see Fig.2b), which enters parametrically through a common effective mass in the respective equation of each mode.

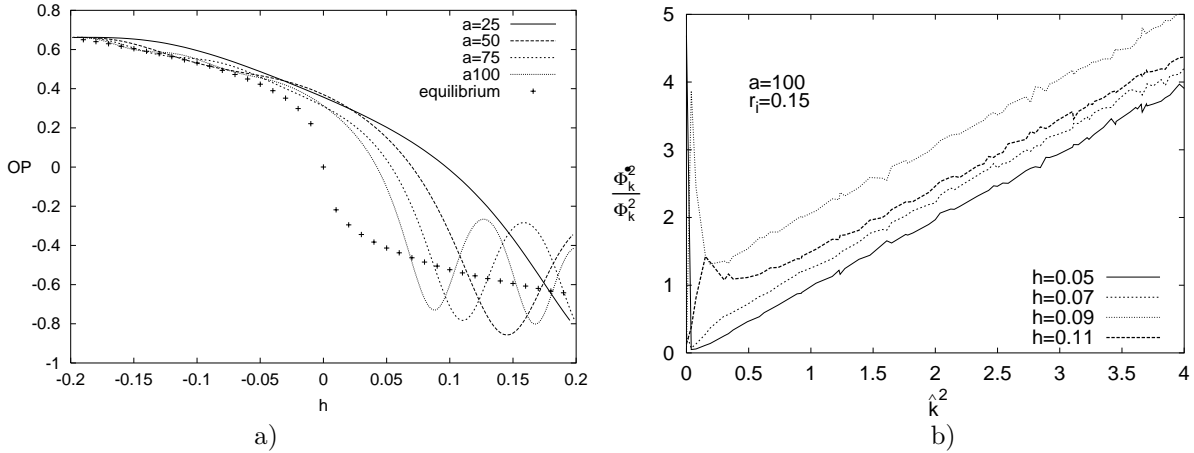


FIG. 2. a) OP trajectories for different $dh/dt = a^{-1}$ values ($L = 64, r_{initial} = 0.15$). The equilibrium $\bar{\Phi}(h)$ curve is represented by crosses. b) Temporal variation of the left hand side of (9) synchronized by the OP oscillation.

One may define the error of OP trajectories from different thermal initial conditions as the standard dispersion of the $OP(t)$ values at fixed t . This error comes out at ≈ 0.007 for $h < 0$ and ≈ 0.02 for $h > 0$, which is about 100 times smaller than the amplitude of the oscillations.

For comparison the equilibrium OP values are also displayed in Fig. 2a. These values come from thermal solutions of Eq.(1) with the (r, h) values chosen from the equilibrium points of Fig.1.

The OP-evolution displayed in Fig.2a is by itself a challenge seeking quantitative interpretation. We shall elaborate on it in detail in Section 4.

IV. HOW LARGE THE CORRELATION LENGTH GROWS?

The main physical motivation of our investigation was to answer the question in the title of this section. We have used the fast method described in Section 2 for deducing the actual non-equilibrium value of the correlation length, that is the inverse σ -mass.

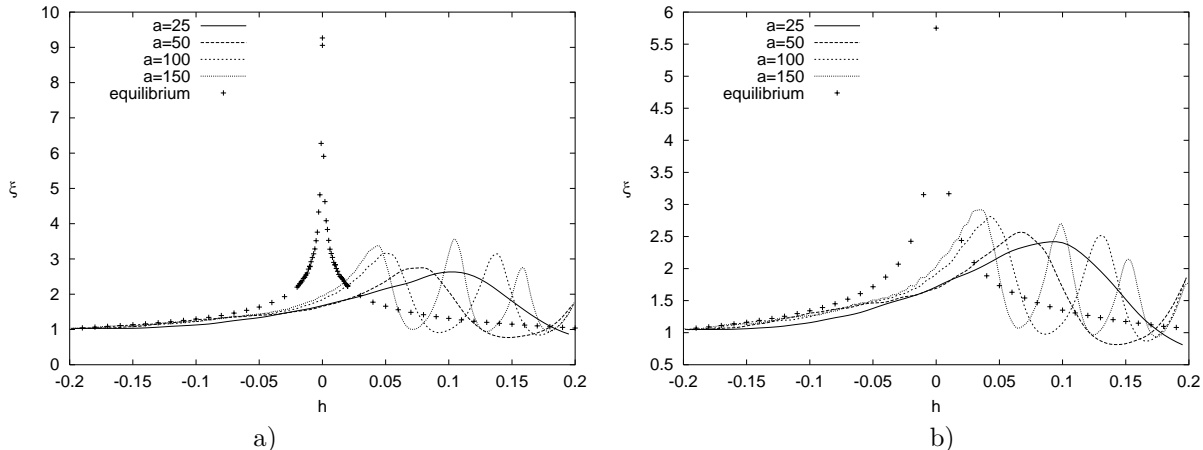


FIG. 3. Evolution of correlation length during a crossover compared to the equilibrium values along the (r, h) route depicted in Fig.1. ($L = 64$, $r_{i,a} = 0.083$, $r_{i,b} = 0.15$). Crosses give the results of equilibrium measurements.

In Fig.3a we display the h -history of the correlation length obtained on a 64^3 lattice with an initial reduced (Ising) temperature $r_i = 0.083$ for four values of the a parameter. The (r, h) trajectory starting from this r_i passes the closest by the critical end point ($r_{min} = -0.0056$), see the lower curve in Fig.1. Again, the equilibrium correlation length values obtained from analyzing long thermal time evolutions with an identical method, are displayed in the same plot.

When compared with the estimate of [6] the most dramatic difference is obvious: even the slowest variation of h considered in BR is in reality much too vehement. The critical slowing down of the internal interactions pushes the system far out of the equilibrium state corresponding to the actual magnetic field value. This is the reason that after passing the maximum of the correlation length, reached with the expected “supercooling” in h , ξ starts to oscillate and the correlation length very steeply drops to a minimum. This oscillation has opposite “phase” if compared to the (Ising) temperature. A probable explanation to this phenomenon is, that a shrinking correlation length means larger σ -mass and hence an increase in the frequency of the microscopic oscillations of the UV-modes. The energy of a weakly coupled UV mode — like of a tuned linear oscillator — gets larger with the increase of its frequency.

The qualitative picture is the same in a wider neighbourhood of the critical end-point as one can see from Fig.3b which corresponds to a trajectory passing somewhat farther ($r_i = 0.15$, the upper curve in Fig.1).

The period of the oscillation in QCD temperature based on the $T_{QCD} - h$ correspondence proposed in [6] seems to be much smaller than the spread of the freeze-out temperature estimates appearing in the literature. Therefore our main qualitative result is that the expected increase in the coherence of the pion radiation might be missed if the actual freeze-out would happen at a temperature slightly beyond the maximum of the correlation length. Ideally, accurate measurements of the freeze-out temperatures of different meson species coupled to σ might allow to map out the variation of its correlation length across the hypersurfaces of the respective “last scatterings” as predicted by our analysis.

One can compare the exact non-equilibrium $\xi(h, a)$ function to the estimate of BR in some more quantitative detail. The maximal values of the correlation length in units of the initial ξ_0 are in the range $(2.5 - 3.5)\xi_0$. The amount of supercooling is generically larger from the numerical solution of the Φ^4 dynamics, in comparison to the result of the first order dynamics conjectured in [6].

A good measure of the amount of the physical supercooling as a function of the h -velocity is offered by the shift in the location of the first maximum of the correlation length, $h(\xi_{max})$. This value turns out to scale with the velocity of the h -variation:

$$h(\xi_{max}, a) = ca^{-0.5 \pm 0.01}. \quad (11)$$

The values of ξ_{max} themselves follow also a scaling form as suggested in [6] on the basis of dynamical scaling considerations. We find numerically

$$\xi_{max}(a) = c' a^{0.211 \pm 0.01}. \quad (12)$$

The value of this exponent agrees with the prediction of BR, when it is applied to the class A. Then their prediction for the exponent yields $\nu/\beta\delta/(1+z\nu/\beta\delta) = 0.215$, with $\nu = 0.630$, $\beta = 0.326$, $\delta = 4.8$ and $z = 2 + (6 \ln(4/3) - 1)\eta$, $\eta = 1/4$ [5]. When the initial (Ising) temperature is tuned to approach the critical point the closest by 2-3% from above, it turns out that the form of the $\xi(h)$ function qualitatively remains the same as described above. The ξ values at the first maxima are about four times larger than those taken in the first minima following them.

The second maxima of ξ appearing in our numerical solution very probably cannot be observed experimentally, since the fireball breaks up into non-interacting mesons before reaching the corresponding low temperature.

The sensitivity of the results to the size of the system is also an important issue, since at present the linear size of the plasma droplet is estimated to about $6\xi_0$. Since we have chosen ξ_0 for the lattice constant, one might expect to reach the maximal allowed correlation length in a much smaller lattice volume already. We have tested the robustness of the above conclusions by varying the lattice size between 16 and 64. No important variation was seen when changing the size from $L = 64$ down to $L = 32$, but a 20% drop in the maxima of ξ appears when going down to $L = 16$.

The final question to be discussed in this section is to what extent the observed features of the time dependence of ξ depend on the algorithm of its determination. In Fig.4 a typical time evolution is displayed for the three characteristic lengths introduced in Section II. Using the normalisation to the common equilibrium mass we see very good agreement of all three before the system is slowed out of equilibrium. Next all three enter an oscillatory regime, with about the same amplitude, but with a certain “phase shift”.

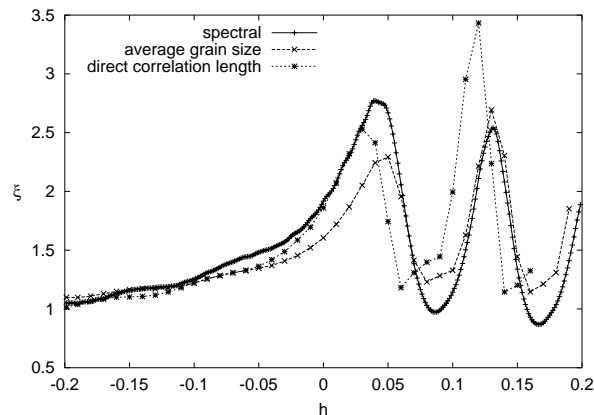


FIG. 4. Comparison of the time dependence of the correlation lengths determined by three different methods: the “direct” correlation length, the average grain size and the inverse mass determined with spectral algorithms. Shown is the evolution during the non-thermal crossover passing close to the Ising critical end-point. ($L = 32$, $r_i = 0.15$, $a = 100$, for definitions see the text.)

This observation makes it more difficult to relate the location of the first maxima in h to the freeze-out temperature. One has to work with that length which is coupled to the microscopical mechanism producing a certain observable. For example, the high frequency part of the σ -field is probably well-represented by a gas of particles with the effective frequency determined by the “susceptibility dispersion” method 5. The decay products of these hard particles will reflect this frequency. The softest part of the pion spectra probably comes from the coherent decay of the longest wavelength components of the field configuration, therefore a characteristic coherence length closer to the “grain size” (method 2) will be followed.

V. THE EFFECTIVE ORDER PARAMETER DYNAMICS

In statistical physics noisy first order effective equations are used for the longest wavelength hydrodynamical modes to describe relaxation phenomena. These equations have to be written down for the order parameter fields and also for composite objects which correspond to densities of conserved quantities [5]. From this viewpoint it is not clear what could be the foundation for the proposal of BR to write down directly an equation for the relaxation of the inverse correlation length.

We follow the more conventional path and discuss first the effective dynamics of the order parameter. After presenting considerable evidence for the validity of our approach to OP, we shall be able to build upon it also a quite natural interpretation of the observed behaviour of the correlation length.

It is obvious that one cannot account for the oscillatory behaviour of the order parameter experienced after passing near the critical end-point just using a first order effective dynamics. This means, that our system actually leaves the hydrodynamical regime. Therefore, we propose to complete the effective OP-equation with an “acceleration” term. For the effective free energy we use the simplest quadratic form which corresponds to an oscillator potential centred at the equilibrium value of the order parameter [7]. We emphasize that no noise is introduced.

The proposed linear equation is of the form

$$a^{-2}\ddot{\Phi}(h) + a^{-1}\Gamma(h')\dot{\Phi}(h) + m_{eq}^2(h')(\bar{\Phi}(h) - \bar{\Phi}_{eq}(h')) = 0. \quad (13)$$

In this fully deterministic equation a “dot” means derivation with respect to h . In view of the relation $h = at$, the derivatives originally refer to the time. The determination of the h -dependent coefficients $m_{eq} = \xi_{eq}^{-1}$, $\bar{\Phi}_{eq}$ follows the methods described in previous sections. However, their values are taken not at the actual h , but at a somewhat smaller value h' , which corresponds to an earlier equilibrium state. This is the simplest way to incorporate the “slowing out of equilibrium” phenomenon into the proposed equation.

The shift $h - h'$ is established by optimizing the agreement with the measured order parameter trajectory. Three physical pictures for this shift were tested and benchmarked by the MS deviation of the reconstructed OP trajectory from the measured one (δ^2). In the first one a global delay parameter is introduced which acts with equal strength before and after reaching the neighbourhood of the critical point ($\delta^2 = 0.0034$). In the second version one switches on the h -delay only when $h \geq 0$ ($\delta^2 = 0.094$). In the third version the delay grows linearly from 0 to a final value which was fitted to the data. This latter method produced the smallest MS deviation ($\delta^2 = 0.0029$), therefore below we shall present results obtained with this method. The representative δ^2 values refer to a 32^3 lattice with $r_i = 0.15$, $a = 100$. As expected the fitted slope of the linear shift, $h - h' = \text{const.} \times (h - h_0)$ clearly increases as r_i gets smaller. The value of the constant coefficient changes monotonically from 0.02 to 0.1 while r_i varies from 0.2 to 0.1.

The reconstruction of the measured OP-trajectory turned out to be rather insensitive to the value of the relaxation rate $\Gamma(h')$. Furthermore, no h' dependence could be observed. Chosen in the wide range $\Gamma = 0.01 \dots 0.6$, an acceptable agreement of the real and the reconstructed OP-trajectory was found.

A nonperturbative, near equilibrium determination of Γ was attempted by stopping h at a certain value h_{stop} and fitting the relaxation of $\bar{\Phi}$ towards equilibrium to an exponential rate. The estimate for Γ was found to be in the range $0.005 \dots 0.025$, independently of r_i and a . In a model investigated previously we found that during the equilibration the relaxation rate approaches its perturbative value strictly from below [10].

Eq.(13) was solved with the initial values for the OP deduced from the full system: $\bar{\Phi}(h = -0.2)$, $\dot{\bar{\Phi}}(h = -0.2)$. OP trajectories resulting from different realizations of the initial thermal ensemble give slightly different initial conditions for Eq. (13). This uncertainty sets the error of the reconstructed trajectory. (The average variance of the solution of Eq. (13) is $\sigma_{th}^2 \approx 0.0001$ for $h < 0$ and $\sigma_{th}^2 \approx 0.0006$ for $h > 0$).

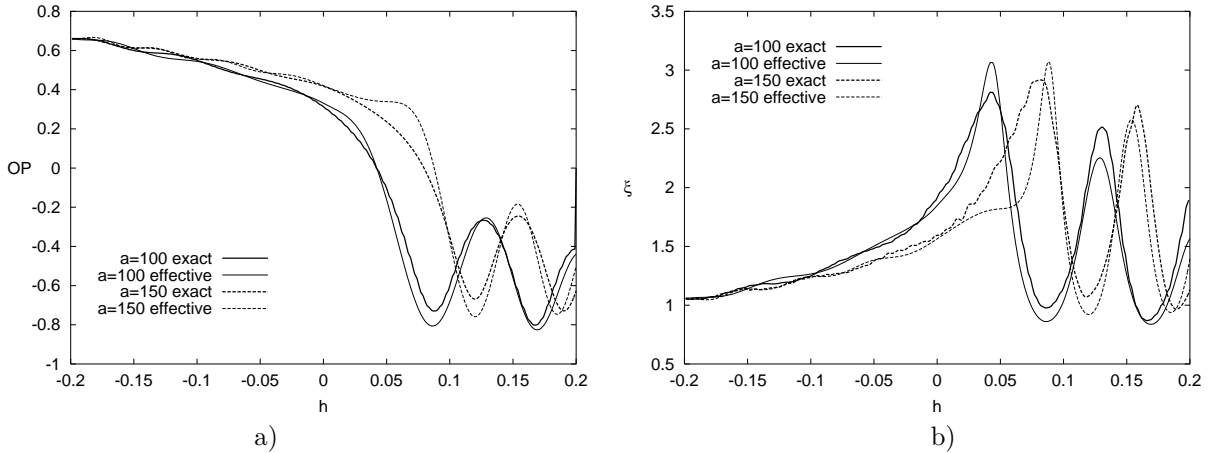


FIG. 5. a) Exact evolution of OP and the solution of Eq.(13) for two values of a . b) Exact evolution of the correlation length and its values estimated with help of the effective OP-trajectory (see text) for two values of a . ($L = 64$, $r_i = 0.15$)

In Fig. 5a we compare the true and the reconstructed OP-trajectories. The quality of the agreement somewhat fluctuates, but its δ^2 value is less than 0.02 in the whole $L, T, a \geq 50$ -region, considered in this paper. In principle one could attempt to improve further the analytic interpretation of the OP-dynamics by introducing a memory kernel into the equation of $\bar{\Phi}$, but we believe that our proposed equation captures the essence of the actual OP-dynamics.

With this achievement, we can return to the discussion of the non-trivial variation of the correlation length. We build our description on our understanding of the order parameter dynamics. This is made possible, by observing a simple functional correspondence realized implicitly between ξ and $\bar{\Phi}$ in the exact time evolution of the system. The functional relation is unique during the regime of smooth OP-evolution; in the oscillatory regime the function can

be extracted only by averaging over several passes. The parameters of the interpolating polynomial turn out to be slightly a -dependent, if $a \leq 100$. Inverting the polynomial one may introduce the piecewise unique function $d\bar{\Phi}(\xi)/d\xi$, which can be used to rewrite the effective equation of $\bar{\Phi}$ into an equation for $\xi(h)$. The agreement of the $\xi(h)$ function reconstructed in this way with the measured (oscillatory) inverse masses is quite spectacular (see Fig.5b).

We conclude this section by discussing a more “theoretical” approach to the determination of Γ, m_{eq} and of Φ_{eq} , by observing that these quantities refer to (near) equilibrium situations. In our very simple model their nonperturbative values were easily determined from the microscopical data. In case of more realistic models, however, one may attempt to use perturbative estimates for the masses, the damping rates and the equilibrium order parameter. We have tested the quality of the reconstructed OP-trajectory in the present model also when these coefficients were taken from perturbation theory.

As it is well known even a resummed perturbation theory fails in the vicinity of the critical temperature due to its bad behaviour in the IR regime. The divergence of the correlation length and as a consequence also of the effective expansion parameter $\lambda T\xi$ (see for example [11]) excludes its use in the scaling regime. In a finite system, however, with an IR cut-off L we could attempt to extract the equilibrium mass, the equilibrium OP value and the damping rate of the OP using two-loop lattice perturbation theory. The mass comes from the second derivative of the effective potential at the minimum, the OP is the location of this minimum, and for the damping rate we use the formula derived in [12]. The two-loop perturbative effective potential was computed with self-consistent propagators on a finite lattice and we extracted from it the position of the minimum and the second derivative in this point along the (r, h) route shown in Fig.1. The mass used in the propagators was determined from a one-loop gap equation.

The expression of the two-loop effective potential on lattice reads as

$$V_{2-loop}(\bar{\Phi}) = V_{tree}(\bar{\Phi}) + \frac{T}{L^3} \sum_{\mathbf{k}} \log \frac{\omega(\hat{\mathbf{k}}, \mu)}{T} - \frac{\lambda T^2}{8L^6} \left[\sum_{\mathbf{k}} \frac{1}{\omega^2(\hat{\mathbf{k}}, \mu)} \right]^2 - \frac{\lambda \bar{\Phi}^2 T^2}{12L^6} \sum_{\mathbf{k}} \sum_{\mathbf{p}} \sum_{\mathbf{q}} \frac{\delta^{(3)}(\hat{\mathbf{k}} + \hat{\mathbf{p}} + \hat{\mathbf{q}})}{\omega^2(\hat{\mathbf{k}}, \mu) \omega^2(\hat{\mathbf{p}}, \mu) \omega^2(\hat{\mathbf{q}}, \mu)}, \quad (14)$$

where $\omega^2(\hat{\mathbf{k}}, \mu) = \hat{\mathbf{k}}^2 + \mu^2$. The mass was determined from the one-loop gap equation

$$\mu^2 = m^2 + \frac{\lambda}{2} \bar{\Phi}^2 + \frac{\lambda T}{2L^3} \sum_{\mathbf{k}} \frac{1}{\omega^2(\hat{\mathbf{k}}, \mu)}. \quad (15)$$

In view of eq.(2) we put $\lambda = 6$.

The inaccuracy of the two-loop perturbation theory comes overwhelmingly from the fact that for the lattice spacing considered in this paper, the estimate of the critical temperature exceeds its actual value by 25%. (We note that if one starts with a higher cut-off (e.g. $a|m| = 0.5$) the agreement improves to 5%.) Despite all inaccuracies of the perturbation theory the solution of Eq.(13) based on the perturbative potential V_{2-loop} turns out to follow quite closely the real trajectory, though because of the ill-determined OP values the deviation has also a systematic error ($\delta^2 \approx 0.014$).

For a second estimate of $\xi_{eq}(h)$ we used Widom’s scaling form [13] like it was done by BR. This yields $\xi_{eq}(h)$ values close to the measured equilibrium correlation length. The solution of Eq.(13) fitted to the measured curve with this coefficient is only of slightly lower quality than the fully nonperturbatively reconstructed one ($\delta^2 \approx 0.0061$).

VI. CONCLUSIONS

In this paper we have presented a detailed discussion of the real time non-equilibrium evolution of the classical Φ^4 field theory when it passes nearby the critical end point in its (r, h) phase diagram under the influence of a time dependent external magnetic field. The numerical investigation was focused on the variation of the correlation length. A quick but accurate method for determination of the instant non-equilibrium mass of the Φ -field was employed, relying on the parametrical relation of the spatial domain of coherence of field fluctuations to the mass.

For a wide range of the rate of variation of the magnetic field we have experienced a slowing out of the system from the hydrodynamical regime. This phenomenon demonstrated itself in the behaviour of all quantities used for the characterisation of the system: the average kinetic energy per site, the order parameter and the correlation length all showed oscillations when the value of the external magnetic field moved beyond the point of “supercooling”.

A simple effective equation was shown to describe this dramatic feature of the OP-evolution. The coefficients of the second order differential equation written down for the order parameter take their values from equilibrium. They can be determined from separate simulations, but also perturbative estimates led to reasonable description of the OP-evolution. A more important feature of the effective equation is that the coefficient functions should be computed for values of the external magnetic field corresponding to some earlier stage of the evolution. The gradual increase in this shift reflects the cumulative effect of the critical slowing down in the proposed effective description.

The motivation for this investigation came from its possible relevance to the σ -meson dynamics in high energy heavy ion collisions [1]. The weakest point in taking over the above features to QCD is the lack of a quantitatively accurate mapping between QCD and the Φ^4 -model in the (T, μ) -plane. Still, our results can be compared in a useful way with the scenario put forward in [6] for the evolution near a hypothetical critical QCD end point.

In the context of heavy ion collisions, it would be interesting to see the effect of a coherently oscillating long wavelength σ -background with variable correlation length, on transversal jet quenching.

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